Discussion by H. S. Uhler, Yale University.

In order to introduce cyclic order we shall interchange two of the coefficients in the second equation and rewrite it as

$$(1 - y)(b_1 + b_2z + b_3x) = g.$$

Obviously, this modification will have no essential influence on the final result.

Solving the third equation for z we obtain

$$z = 1 - \frac{h}{c_1 + c_2x + c_3y}.$$

Substituting this expression for z in the first equation and performing elementary reductions, we find

$$\alpha y^2 + \beta y + \gamma = 0 \tag{1}$$

where

$$\alpha \equiv a_2c_3(1 - x),$$

$$\beta \equiv [c_3(a_1 + a_3) + a_2c_1 - c_3d] - [c_3(a_1 + a_3) + a_2(c_1 - c_2)]x - a_2c_2x^2,$$

$$\gamma \equiv [c_1(a_1 + a_3) - c_1d - a_3h] - [(c_1 - c_2)(a_1 + a_3) + c_2d - a_3h]x - c_2(a_1 + a_3)x^2.$$

The second equation (modified), when treated in the same manner, gives

$$Ay^2 + By + C = 0 \tag{2}$$

where

$$A \equiv c_3[(b_1 + b_2) + b_3x],$$

$$B \equiv [(c_1 - c_3)(b_1 + b_2) + c_3g - b_2h] + [c_2(b_1 + b_2) + b_3(c_1 - c_3)]x + b_3c_2x^2,$$

$$C \equiv -\{[c_1(b_1 + b_2) - c_1g - b_2h] + [c_2(b_1 + b_2) + b_3c_1 - c_2g]x + b_3c_2x^2\}.$$

Eliminating y from equations (1) and (2) we get

$$(\alpha C - \gamma A)^2 - (\alpha B - \beta A)(\beta C - \gamma B) = 0. \tag{3}$$

When the expressions for α, β, γ, A, B, C are substituted in relation (3) it is found that the coefficient of x^6 vanishes identically, but that all the remaining coefficients have extremely complicated values other than zero, in general. Therefore, since the equation in x is of the fifth degree with literal coefficients, the original set of equations cannot be "solved."

In the particular case where $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $b_1 = 5$, (new) $b_2 = 6$, (new) $b_3 = 7$, $c_1 = 8$, $c_2 = 9$, $c_3 = 10$, $d = 11$, $g = 12$, and $h = 13$, I found

$$1{,}628{,}991x^5 + 3{,}399{,}760x^4 - 3{,}151{,}759x^3 - 8{,}498{,}412x^2 - 8{,}179{,}912x - 8{,}133{,}008 = 0,$$

which has at least one positive real root.

411 (Algebra) [April, 1914, June, 1919]. Proposed by V. M. SPUNAR, Chicago, Ill.

Determine $x_1, x_2, x_3, \cdots x_p$, from the equations:

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_p &= a_0, \\ b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_px_p &= a_1, \\ b_1^2x_1 + b_2^2x_2 + b_3^2x_3 + \cdots + b_p^2x_p &= a_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots \\ b_1^{p-1}x_1 + b_2^{p-1}x_2 + b_3^{p-1}x_3 + \cdots + b_p^{p-1}x_p &= a_{p-1}. \end{aligned}$$

Solution by H. S. Uhler, Yale University.

From the theory of determinants, we know that

$$x_i = \begin{vmatrix} a_0 & 1 & \cdots 1 & 1 & \cdots 1 \\ a_1 & b_1 & \cdots b_{i-1} & b_{i+1} & \cdots b_p \\ a_2 & b_1^2 & \cdots b_{i-1}^2 & b_{i+1}^2 & \cdots b_p^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{p-1} & b_1^{p-1} & \cdots b_{i-1}^{p-1} & b_{i+1}^{p-1} & \cdots b_p^{p-1} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & \cdots 1 & 1 & \cdots 1 \\ b_i & b_1 & \cdots b_{i-1} & b_{i+1} & \cdots b_p \\ b_i^2 & b_1^2 & \cdots b_{i-1}^2 & b_{i+1}^2 & \cdots b_p^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_i^{p-1} & b_1^{p-1} & \cdots b_{i-2}^{p-1} & b_{i+1}^{p-1} & \cdots b_p^{p-1} \end{vmatrix} \equiv \frac{D_1}{D_2}$$

Consider the determinant in the denominator, D_2. If any b were equal to any one of the $p - 1$ remaining b's, two columns would be identical and D_2 would vanish. Hence,

$$D_2 = k\Delta(b_i - b_1)(b_i - b_2) \cdots (b_i - b_{i-1})(b_i - b_{i+1}) \cdots (b_i - b_p),$$

where $\Delta \equiv (b_1 - b_2) \cdots (b_1 - b_p)(b_2 - b_3) \cdots (b_2 - b_p) \cdots (b_{p-1} - b_p)$, and k can only be a non-literal factor. To determine k, we note that the product of all the first terms in the $\frac{1}{2}p(p - 1)$ binomial factors equals $b_i^{p-1}b_1^{p-2}b_2^{p-3} \cdots b_{p-2}^2 b_{p-1}$ which differs only in sign from the algebraic value of the negative diagonal of D_2; hence, $k = -1$ or

$$D_2 = -\Delta(b_i - b_1)(b_i - b_2) \cdots (b_i - b_{i-1})(b_i - b_{i+1}) \cdots (b_i - b_p). \quad (1)$$

Next, consider D_1. If $a_0, a_1, a_2, \cdots, a_{p-2}, a_{p-1}$ were replaced, respectively, by $1, x, x^2, \cdots, x^{p-2}, x^{p-1}$ we should have (by the same argument)

$$D_x = -\Delta(x - b_1)(x - b_2) \cdots (x - b_{i-1})(x - b_{i+1}) \cdots (x - b_p)$$
$$\equiv -\Delta(x^{p-1} + c_1x^{p-2} + c_2x^{p-3} + \cdots + c_{p-2}x + c_{p-1}).$$

From the theory of equations, we know that $-c_1$ is the sum of $b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_p$, that $+c_2$ is the sum of the products of these b's taken two at a time, etc., and that $\pm c_{p-1}$ is the product of all these b's, the upper or lower sign to be taken according as p is odd or even. Now the first minors that multiply the a's in D_2 are identical with the corresponding minors which multiply the various powers of x in D_x. Consequently,

$$D_1 = -\Delta(a_{p-1} + c_1a_{p-2} + c_2a_{p-3} + \cdots + c_{p-2}a_1 + c_{p-1}a_0). \quad (2)$$

Finally, dividing equation (1) by equation (2) we get the required formula

$$x_i = \frac{a_{p-1} - (\Sigma b)a_{p-2} + (\Sigma bb)a_{p-3} + \cdots \pm (\Pi b)a_0}{\Pi(b_i - b)},$$

the interpretation of the notation being clear from the preceding discussion.

Also solved by NORMAN ANNING.

455 (Geometry) [February, 1915; June, 1919]. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

SOLUTION BY R. C. ARCHIBALD, Brown University.

This problem is closely allied to the problem: Find the maximum triangle of assigned angles circumscribed to a given triangle. Both problems were proposed for solution in 1811 on the last page (exclusive of index), 384, of the first volume of *Annales de mathématiques pures et appliquées*. Solutions by Rochat, and others, were given in volume 2, pages 88–93.

Let $D_1E_1F_1$ be the triangle of assigned angles and ABC be the given triangle. On CA (on the side opposite from B) describe an arc of a circle containing an angle equal to the angle E_1. On CB (on the side opposite from A) describe an arc of a circle containing an angle equal to D. Then if through C a line is drawn parallel to the line of centers of the circles it will meet the circles again in D and E such that DE is the greatest of all line-segments drawn through C. Then if EA, DB be produced to meet in F, the triangle DEF will clearly be the maximum triangle circumscribing the triangle ABC, and the triangle ABC will be the minimum triangle inscribed in the triangle DEF.

Hence to solve the given problem find the maximum triangle, $A_1B_1C_1$, similar to the triangle ABC and circumscribing the triangle $D_1E_1F_1$ (the point F_1 falling in A_1B, and the point E_1 in A_1C_1; also the angles A_1, B_1, C_1 correspond respectively to A, B, and C). Then divide AB at F in the ratio $A_1F_1 : F_1B$; also AC at E in the ratio $A_1E_1 : E_1C$; and CB at D as $C_1D_1 : D_1B_1$. Then D, E, and F are the vertices of the required minimum triangle.

In general, there are, as Rochat pointed out, six solutions of the problem, depending upon the arrangement of the vertices of the triangle DEF on the sides of the given triangle.

Another solution is given in *A Key to the Exercises in the First Six Books of Casey's Elements of Euclid* by Joseph B. Casey, second edition, 1887, pp. 120–121.

Note. A solution by John Casey is given in his *Sequel to . . . the Elements of Euclid*, second edition, 1882, pp. 38–39.—Editors.

Also solved by A. PELLETIER.